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SHOCK WAVES IN A WEAKLY ANISOTROPIC ELASTIC INCOMPRESSIBLE MATERIAL†

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The effect of anisotropy on shock waves in an incompressible elastic medium is studied. No assumption is made here concerning the smallness of the deformations. Particular attention is paid to those materials for which the non-linear stress–strain relation has a point of inflection. The anisotropy is assumed to be small compared with the effect of the non-linear properties of the medium, but this turns out to be quite sufficient for new qualitative effects to be revealed.

A set of states behind the shock wave (the shock adiabatic curve) for a specified state ahead of the shock wave as well as the velocity of motion of the shock wave front for an incompressible medium with small anisotropy of arbitrary form are found and investigated using conservation laws. The shock adiabatic curve is clearly presented in the case of the simplest actual form of anisotropy. The segments on it which correspond to the requirements of an evolutionary character and no decrease in the entropy are indicated.

1. In order to describe the motion of an elastic medium with plane waves, we shall use a Lagrangian Cartesian system of coordinates x_i $(i=1, 2, 3)$ of the initial state where the axis $x_3=x$ is taken as being along the normal to the wave front while the axes x_α $(\alpha=1, 2)$ are in the plane of the wave front. In the general case, a deformation in the Lagrangian description is characterized by the displacement gradient tensor $\partial w_i/\partial x_j$. Only two components of the above-mentioned tensor, that is, $\partial w_i/\partial x$, $i=1, 2$ $(\partial w_3/\partial x\equiv 0$ on account of the incompressibility of the medium), can change during the passage of a plane wave in an incompressible medium, for which we shall adopt the notation $\partial w_i/\partial x=u_i(x, t)$.

It is well known that a hyperbolic system of equations of motion of an elastic medium [1] admits of discontinuous solutions. Their occurrence may be due to discontinuous initial and boundary data and they may also arise when the wave profile is deformed during its evolution. The study of Riemann plane waves in an elastic solid [2] enables one to indicate, in the case of each real material, those processes which lead to an inversion of the wave profile and to the formation of a discontinuity.

On a surface of discontinuity, relationships, which follow from the integral laws for the conservation of mass, momentum and energy, must be satisfied. We shall denote the magnitude of the discontinuity in any physical quantity at the shock front by $[a]=a^+-a^-$, where a^- is the value of the physical quantity immediately ahead of the discontinuity and a^+ (or a) is the value immediately behind it. If at the front there is no generation or absorption of mass, momentum and energy, then the conditions at a discontinuity in an elastic medium have the form [1, 3]

$$\left[\frac{\partial\Phi}{\partial u_i}\right]=\rho_0W^2[u_i],\quad [\Phi]-\frac{1}{2}\left(\frac{\partial\Phi}{\partial u_i}+\frac{\partial\Phi^-}{\partial u_i}\right)[u_i]=0,\quad i=1,2,3 \tag{1.1}$$

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Here W is the velocity of propagation of the wave front along the Lagrangian coordinate, $\Phi(u_i, S)$ is the elastic potential of the medium (the internal energy per unit of initial volume), S is the entropy per unit mass and ρ_0 is the density of the medium in the initial state. In the case of an incompressible material, $i = 1, 2$ and $\rho_0 = \rho = \text{const}$. For a specified state ahead of the shock wave, system (1.1) defines a set of possible states behind the shock wave, each of which corresponds to its own value of W. This single-parameter set represents a curve in phase space, u_i, S which is referred to as the shock adiabatic curve. It is obvious that its form and properties depend on the actual function $\Phi(u_i, S)$ which specifies the material.

In the case of a compressible medium, the possible sates behind a low-intensity shock wave in an unstressed isotropic medium have been investigated previously in [1]], those in the case of a previously deformed or naturally anisotropic medium have been investigated in [3, 4] and the possible states behind a shock wave of finite intensity in an isotropic material with a special form of the function Φ have been investigated in [5, 6]. In the case of an isotropic incompressible medium, the plane polarized shock waves (in which only one of the quantities u_1, u_2 changes) have been studied in [7].

In isotropic media, the elastic potential Φ depends, in a symmetric manner, on the components u_1 and u_2 of the shear deformations in planes parallel to the wave front. Then, instead of the components u_1 and u_2, one can introduce the modulus of shear deformation ε_τ using the equality $\varepsilon_\tau^2 = u_1^2 + u_2^2 \equiv r^2$. By virtue of the assumed smoothness of the dependence of Φ on u_1 and u_2 in an isotropic material, $\Phi = \Phi(r^2, S)$. The internal energy of a medium with an electromagnetic field frozen into it in magnetohydrodynamics possesses a similar property [8]. Also, in anisotropic media, the direction of propagation of plane waves can turn out to be such that there is symmetry in the dependence of Φ on u_1 and u_2, that is, $\Phi = \Phi(r^2, S)$ again, in the plane of the wave front. We shall refer to such a situation as "wave isotropy" and subsequently refer to the case which differs from this as anisotropy.

Below, we study shock waves in incompressible media in the case of which we assume that the divergence from wave isotropy is small and we represent the elastic potential in the form of two terms

$$\Phi(u_i, S) = F(u_1^2 + u_2^2, S) + gp(u_i) \tag{1.2}$$

the first of which contains all the basic non-linear properties of the medium while the second adds a small anisotropy, where $p(u_i)$ is a function of arbitrary form and $g > 0$ is a small scale factor. Apart from this, we shall assume that the dependence of Φ on the entropy S is additive. The elastic potential is then given by the expression

$$\Phi(u_i, S) = F(r^2) + gp(u_1, u_2) + \psi(S), \quad r^2 = u_1^2 + u_2^2 \tag{1.3}$$

The calculation of the stresses from the strains is determined by the derivatives of $\Phi(u_i, S)$ with respect to u_i when $S = \text{const}$. The assumption regarding the additivity of the entropy in the study of the shock waves means that a change in the entropy in these waves has only a negligibly small effect on the dependence of the stresses on the strains. In particular, this assumption holds when the change in the entropy in a shock wave is small which, as will become clear subsequently, is satisfied quite well in the case of those shock waves whose behaviour is substantially affected by a small anisotropy.

In the absence of no anisotropy ($g = 0$), the dependence of the shear stress σ_τ on the shear strain $\varepsilon_\tau = r$ is given by the formula

$$\sigma_\tau = [(\partial F / \partial u_1)^2 + (\partial F / \partial u_2)^2]^{1/2} = dF / dr$$

Let us use the notation $dF/dr = f(r)$. In certain cases, it is convenient to assign negative values to r, assuming that $F(r)$ = an even function. Then $f(r)$ is an odd function. The investigation of low-intensity shock waves has shown that there is a substantial difference in their behaviour depending on the direction of convexity of the graph of $f(r)$, which represents the link between the shear stresses and the shear strain. The formation of shock waves from Riemann waves has also pointed to similar differences [2]. In order to encompass a wider class of media which allow of large elastic deformations, let us take a function $f(r)$ which changes the direction of convexity, which is shown in Fig. 1, for example. Such a form of the dependence $f(r)$ with a point of inflection is observed experimentally, for example, in materials of the rubber type. These materials are characterized by an almost complete absence of bulk strains [9, 10]. Then, the point of inflection of the graph of $f(r)$ is denoted by $r = r^{\sim}$.

If a ray, which passes from the origin of coordinates through a point corresponding to a state behind the discontinuity, intersects the graph of the function $f(r)$ twice, then we shall refer to the point which is closer to the origin of coordinates as A and that which is further from the origin as B. These points (one or the other) will subsequently be adopted as the initial state in the study of discontinuities.

The whole of the investigation can be illustrated in the phase plane of the shear strains u_1 and u_2. For the initial state, we adopt the notation $u_\alpha^- = U_\alpha$, $r^- = R$, $f(R) = f_0$. Equations (1.1), in the case of medium described by the elastic potential (1.3), have the form

$$f\frac{u_\alpha}{r} - f_0\frac{U_\alpha}{R} + g(p_\alpha - p_\alpha^-) = \rho W^2(u_\alpha - U_\alpha)$$

$$[\psi] = -[F] + \frac{1}{2}\left(f\frac{u_\alpha}{r} + f_0\frac{U_\alpha}{R}\right)[u_\alpha] - g\left([p] - \frac{1}{2}(p_\alpha - p_\alpha^-)[u_\alpha]\right) \tag{1.4}$$

$$p_\alpha \equiv \partial p/\partial u_\alpha, \quad R = (U_1^2 + U_2^2)^{1/2}, \quad \alpha = 1,2$$

2. We will first present the properties of the discontinuities which are subsequently required in the case of wave isotropy ($g = 0$). Some of these are well known [5–7] while the others can be readily obtained for the cases which are considered below. By eliminating W from system, (1.4) when $g = 0$, it is possible to obtain the equation of the shock adiabatic curve passing through the initial point

$$\left(\frac{f}{r} - \frac{f_0}{R}\right)(U_1u_2 - U_2u_1) = 0 \tag{2.1}$$

It is obvious that, in the u_1, u_2 plane, the shock adiabatic curve consists of the line $U_1u_2 - U_2u_1 = 0$, which passes through the origin of coordinates and the initial point, and the circles $f(r)/r = f_0/R$ (Fig. 2). There can be two such circles in the case of the form of the function $f(r)$ which has been adopted. They pass through points A and B (Fig. 1), one of which

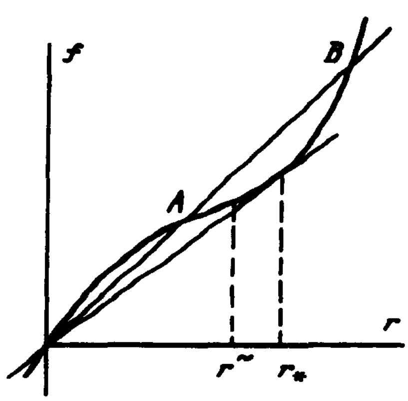

Fig. 1.

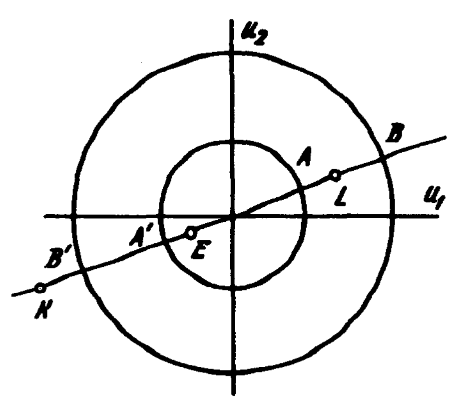

Fig. 2.

represents the initial state. This form of shock adiabatic curve enables one to separate out all of the discontinuities into plane-polarized discontinuities, corresponding to jumps from one point of the straight line onto another, and rotational discontinuities which correspond to jumps from one point of a circle to another point of the same circle and consider them independently [5–8]. For example, if the initial state corresponds to point A, then a jump from this point to some other point on the circle $r = r_B$ is treated as two discontinuities moving with the same velocity: a plane polarized jump from A to B and a rotational discontinuity along the circle $r = r_B$.

$$\rho W^2 = (f - f_0)/(r - R) \tag{2.2}$$

follows from (1.4) for the velocity of plane polarized discontinuities.

If the jump is of very small intensity, then its velocity becomes the corresponding characteristic velocity $\rho W_r^2 = \rho c_r^2 = df/dr$. According to (1.4), the velocity of a rotational discontinuity is $\rho W_\theta^2 = f/r$ and is identical to the characteristic velocity c_θ. A geometrical interpretation can be given to the above-mentioned velocities in the form of the angles of inclination of the chord joining the initial and final point of the graph of $f(r)$ in Fig. 1 (W_r) of the tangent to this graph (c_r) and the ray from the origin of coordinates to the point under consideration ($W_\theta = c_r$)

The difference in the angles of inclination of the secant and tangent at this point is represented by the function $d = f/r - f'$. The fact that it tends to zero in the neighbourhood of $r = 0$ corresponds to the transition to linear elasticity. When $d \neq 0$, we shall use the notation $c_1 = \min\{c_\theta, c_r\}$, $c_2 = \max\{c_\theta, c_r\}$ and refer to the waves corresponding to them as the slow and fast waves, $c_2 > c_1$. In the graph of $f(r)$ (Fig. 1), apart from $r = 0$, a further point $r = r_* > r$ exists such that $d(r_*) = 0$ and $c_\theta = c_r$. To the left of this point $d > 0$ and $c_2 = c_\theta$, that is, the rotational wave is fast. For states when $r > r_*$, the plane polarized wave $c_2 = c_r$ will be fast since $d < 0$.

The requirement that shock waves are of an evolutionary character, which expresses the correctness of the boundary conditions on the shock front, imposes constraints on the velocity of the jump W. The conditions for an evolutionary character of a general type in the case of discontinuities in media with two characteristic velocities which differ in their moduli requires that one of the two systems of inequalities

$$(a)\ (c_2^-)^2 \leqslant W^2,\quad (c_1^+)^2 \leqslant W^2 \leqslant (c_2^+)^2, (b)(c_1^-)^2 \leqslant W^2 \leqslant (c_2^-)^2,\quad w^2 \leqslant (c_1^+)^2 \tag{2.3}$$

should be satisfied.

In case a the discontinuity is said to be fast while, in case b, it is said to be slow. It is obvious that plane polarized and rotational waves can be both fast as well as slow depending on the sign of d.

It was found above that the velocity of a rotational discontinuity $W_\theta = f/r$ is identical to the characteristic velocities c_θ on both sides of a discontinuity. Hence, according to (2.3), a rotational discontinuity is evolutionary. Furthermore, it is readily verified that the entropy does not change at this discontinuity, $[S] = 0$ and the thermodynamic requirement that the entropy should not decrease is satisfied. The above-mentioned properties of a rotational discontinuity make it indistinguishable from a Riemann wave of the same type [2]. For subsequent purposes it is necessary to point out that the two constraining requirements imposed on rotational discontinuities are satisfied in the form of equalities, that is, on the bounds of what is permissible.

In the case of plane polarized waves it is easy to find the parts of (2.23) with the evolutionary property using the geometrical interpretation of the expressions for the velocity of a discontinuity and the characteristic velocities. The position of these parts depends on which of the points A or B is adopted as the initial state.

Those parts of the plot of $f(r)$ which correspond to conditions (2.3) are distinguished by the bold lines in Figs 3(a) and (b): (a) in the case of the initial state at point A and (b) at point B. At the ends of the sections with the evolutionary property, the velocity of the discontinuity is identical to one of the characteristic velocities. By analogy with the theory of detonation, we shall refer to such points as Jouguet points. In Fig. 3(a), $W_A = W_L = c_1^-$, $W_0 = W_{A'} = c_2^-$, $W_E = c_1^+$ at the Jouguet points and, in Fig. 3(b), $W_B = W_K = c_2^-$, $W_0 = W_{B'} = c_1^-$, $W_E = c_1^+$. Point L was obtained at the intersection of the plot of $f(r)$ with the tangent to it at the initial point. The case when $R = r_A < r^\sim$. is shown in Fig. 3(a). If $r_A > r^\sim$, then, as before, the section LA will be non-evolutionary but point L is located to the left of point A.

Apart from the general conditions for the evolutionary character of (2.3), additional conditions must be imposed in the case of wave isotropy ($g = 0$) for plane-polarized discontinuities which involve the satisfaction of a further one of the two systems of inequalities [11]

$$(c)\,(c_\theta^-)^2 \leqslant W^2,\quad (c_\theta^+)^2 \leqslant W^2,\quad (d)\,W^2 \leqslant (c_\theta^-)^2,\quad W^2 \leqslant (c_\theta^+)^2 \tag{2.4}$$

These requirements follow from the fact [8] that, when $g = 0$, the interaction between the plane-polarized discontinuity and small rotational perturbations occurs independently of the interaction with the remaining perturbations and without changing the velocity of the discontinuity. The inequalities (2.4) represent the conditions for the problem of finding the amplitudes of the small rotational-type perturbations which emerge from the discontinuity to be uniquely solvable. It can be shown that conditions (2.4) forbid only discontinuities with a change in the sign of r. This leads to the exclusion of the sections $A'E$, $K'B$ and EO from the solution. However, we shall not register this in Fig. 3 on account of the fact that subsequently, in the general case when $g \neq 0$, the additional inequalities (2.4), which are conditional upon isotropy, and, consequently, the constraints imposed by them, will be absent.

To investigate the change in entropy at the discontinuity, we obtain the following relationship

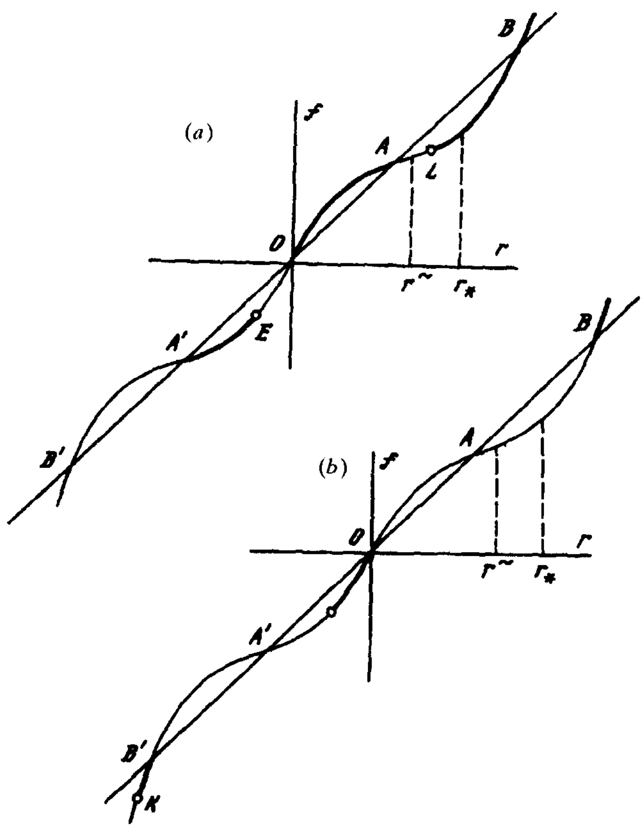

Fig. 3.

$$[\psi(S)] = \frac{1}{2}(f + f_0)(r - R) - \int_R^r f(r)dr \tag{2.5}$$

from (1.4)

The last term on the right-hand side is $[F(r)]$. Since $d\psi/dS = \partial\Phi/\partial S = \rho T > 0$ (where T is the temperature), the sign of $[\psi]$ is identical to the sign of $[S]$. Hence, the condition that the entropy should not decrease at the discontinuity requires that the area included between the segment of the secant passing from the initial point to the final point and the plot of $f(r)$ in the same segment should be positive (Fig. 1). All segments, where conditions (2.3) and (2.4) are simultaneously satisfied, satisfy this requirement.

3. The degeneracy, which is introduced by the isotropy of a medium, mainly manifests itself in the existence of rotational discontinuities in which there is no change in entropy, but the modulus of the strain changes and the velocity of the front, which is the same for all states behind the discontinuity, is identical to the characteristic velocities on the two sides of the discontinuity. All of this leads to the fact that the conditions for the evolutionary property and the requirement that the entropy should not decrease are satisfied for rotational discontinuities in the form of equalities, that is, they are located on the bounds of what is permissible and any small deviation of a discontinuity from being rotational can immediately make it unrealizable. Subsequently, we shall therefore pay most attention to discontinuities which are close to rotational discontinuities in the case of which, on account of anisotropy, one would expect substantial qualitative changes in the composition of the solution.

In the case of plane-polarized waves all of the above-mentioned characteristics are functions of r and the addition of small terms with anisotropy only introduces a small quantitative correction to their values.

Considering the case when $g \neq 0$, we note first of all that, first, the separation of the waves into plane polarized waves and rotational waves loses its meaning and, second, when $g \neq 0$, the additional evolutionary conditions (2.4), associated with the above separation of the waves and with the special form of the dependences when $g = 0$, must not be imposed. A discussion of these questions in the case of small-amplitude waves has been given in [11] and it equally applies to waves of finite amplitude.

Since the case of small values of g is being considered, it may be expected that the shock adiabatic curves of the velocity of the discontinuities, of the small perturbations and of the entropy jumps change only slightly compared with the case when $g = 0$. One may therefore speak of quasi-plane-polarized and quasi-rotational discontinuities, bearing in mind the closeness of the point which depicts the final state to the corresponding part of the shock adiabatic curve when $g = 0$. However, when $g \neq 0$, such a separation loses its strict sense.

We will initially consider the form of the shock adiabatic curve, described by an equation which we obtain from (1.4) by the elimination of W

$$\left(\frac{f}{r} - \frac{f_0}{R}\right)(U_1u_2 - U_2u_1) = gH, \quad H = (p_2 - p_2^-)(u_1 - U_1) - (p_1 - p_1^-)(u_2 - U_2) \tag{3.1}$$

Assuming that g is small in magnitude, it is possible to calculate the values of the right-hand side of H at the corresponding points of the shock adiabatic curve when $g = 0$ and thereby to treat it as being known. In the case of quasi-rotational and quasi-plane-polarized waves, we then obtain respectively

$$\Delta r = g\frac{Hr}{(U_2u_1 - U_1u_2)d}, \quad d = \frac{f}{r} - f', \quad U_1u_2 - U_2u_1 = g\frac{H}{f/r - f_0/R} \tag{3.2}$$

The left-hand sides of these equalities characterize the deviations of the points of curve (3.1) from the shock adiabatic curve which corresponds to $g = 0$, that is, from the circles $r = r_A$ or

$r = r_B$ and from the straight line $U_1u_2 - U_2u_1 = 0$ (Fig. 2). The right-hand sides are calculated on the shock adiabatic curve with $g = 0$. If $H \neq 0$, then, according to (3.1), the calculated deviations tend to infinity as the point of intersection of the circle with the straight line is approached and the shock adiabatic curve itself in the neighbourhood of this point behaves like an hyperbola. This conclusion does not apply to the neighbourhood of the initial point where $H = 0$ and where there is always a point of self-intersection of the branches of the shock adiabatic curve.

The part of the shock adiabatic curve close to the circle $r = R$, which passes through the initial point, represents quasi-rotational discontinuities. Small values of the jumps in the modulus of the strain $r - R \sim g$ and the entropy $S - S' \sim g$ are characteristic of such discontinuities. The branch of the shock adiabatic curve which is close to the second circle $r \neq R$ corresponds to jumps with a change in r and S by a finite amount. Such discontinuities are no longer quasi-rotational discontinuities, although they do have much in common with them. We shall call them quasi-circular discontinuities.

Conditions (2.3), where the velocity of the jump is represented by the expression $\rho W^{22} = (f - f_0)/(r - R) + g\xi$ and the function ξ is bounded as $g \to 0$, must be satisfied when searching for the sections of quasi-plane-polarized discontinuities with evolutionary character on the shock adiabatic curve when $g \neq 0$. In this formula, the first term is the main term, which has been used previously in (2.2) in verifying conditions (2.3) when $g = 0$.. This does not enable one to write out the function ξ in explicit form. The presence of a second term in the expression for W only slightly displaces the ends of the evolutionary sections (Jouguet points) obtained when $g = 0$. As previously, they may be taken to be as shown in Fig. 3. Moreover, the intervals close to the sections $A'E$, KB' and EO, which are excluded by conditions (2.4) when $g = 0$, will also be evolutionary.

In verifying the thermodynamic requirement $[S] \geqslant 0$ for quasi-plane-polarized discontinuities, a small correction of $\sim g$ on account of anisotropy in the rule of areas (2.5) again, in the general case, only slightly displaces the positions of the ends of the sections where $[S] \geqslant 0$ found when $g = 0$ by the geometric method. Here, there is no need to find these intervals exactly and it suffices to check that the condition $[S] \geqslant 0$ is satisfied in just the evolutionary intervals which have been obtained. It turns out that, in all of the intervals shown in Figs 3(a) and (b) apart from EO, the thermodynamic requirement is satisfied and, furthermore, the evolutionary condition (2.3) appeared to be more rigorous than $[S] \geqslant 0$.

The section EO requires a special discussion. The fact that the condition $[S] \geqslant 0$ is satisfied in this interval is determined by the fact that the initial point B is located far from the origin of coordinates in the plot of $f(r)$. depending on this, a thermodynamically appropriate domain around the origin of coordinates may contain point O, it may lie completely to the left of it or disappear completely when point B departs sufficiently far to the right. In this case, the section, where conditions (2.3) and $[S] \geqslant 0$ are simultaneously satisfied, contracts to the point E and, consequently, the thermodynamic requirement turns out to be stronger than (2.3).

In the case of jumps to points of the shock adiabatic curve close to the circles, the velocity W is calculated using a formula obtained from (1.4)

$$\rho W^2 = \frac{f}{r} + g\frac{(p_2 - p_2^-)U_1 - (p_1 - p_1^-)U_2}{U_1u_2 - U_2u_1} \tag{3.3}$$

The difference between W^2 and the velocity of the characteristics f/r and, consequently, the domains where conditions (2.3) are satisfied, are exclusively determined by the second term which introduces the anisotropy. It is also the same in the case of the entropy jump $[S]$ and the difference in this quantity from zero or a constant in the case of quasi-circular discontinuities is solely determined by the anisotropic terms. For this purpose, it is necessary to have the function $p(u_\alpha)$ specified in explicit form.

4. We will now consider a medium with a specific type of anisotropy. Let $p = 1/2(u_2^2 - u_1^2)$. The function p has such a form in the domain of small deformations when there is anisotropy created by preliminary deformation [3] as well as for orthotropic, transversely isotropic and certain other materials [4]. The equation of the shock adiabatic curve (2.1) takes the form

$$\left(\frac{f}{r} - \frac{f_0}{R}\right)(U_1 u_2 - U_2 u_1) = g(u_1 - U_1)(u_2 - U_2) \tag{4.1}$$

It is shown in Fig. 4(a) when the initial state is represented by point A and $R = r_A < r_*$ and, in Fig. 4(b), when $R = r_B > r_*$. Sections close to the straight line $U_1 u_2 - U_2 u_1 = 0$ which serves as an asymptote at $\pm\infty$, correspond to quasi-plane-polarized waves. The difference in the form of the curve depends on the form of the function $f(r)$ and the position of the initial point $r = R$. The deviation from the circles $r = r_A$ and $r = r_B$ is determined by the quantity Δr according to (3.2), where $H = (u_1 - U_1)(u_2 - U_2)$ in the case of the chosen medium. The intersection of the shock adiabatic curve with the above-mentioned circles and a change in the sign of Δr occurs at points with the coordinates $u_1 = U_1$ and $u_2 = U_2$. Here, because of the opposite signs in the case of the function $d = f/r - f'$ on the circles $r = r_A$ and $r = r_B$ and the deviation Δr is in the opposite direction at the corresponding points on these circles.

In calculating the entropy jump, the coefficient accompanying g in formula (1.4) for the chosen form of the function $p(u_\alpha)$ was found to be equal to zero so that, as previously, the change in the entropy in such a medium is found using the rule of areas (2.5). When $R = r_A < r_*$ (Fig. 4a), $d(R) > 0$ and the entropy jump is non-negative within the circle $r = R$, that is, at those points of the shock adiabatic curve for which $|u_2| \geqslant U_2$ To it is added the whole of the domain of the shock adiabatic curve close to the circle $r = r_B$ and the section which departs to $+\infty$ along the asymptote. In the case of an initial state which is specified by point B, $R = r_B > r_*$ (Fig. 4b), and the intervals of the shock adiabatic curve outside of the circle $r = R = r_B$, that is, $|u_2| \geqslant U_2$ satisfy $d(R) < 0$ and the condition $[S] \geqslant 0$. Branches, which depart to $\pm\infty$ along the asymptote are added to it and a certain interval in the neighbourhood of the origin of coordinates, the existence of which has been stipulated above and depends on the form of the function $f(r)$ and how far away the initial point B is on the graph of f.

In order to verify the evolution conditions (2.3) we use the expressions for the characteristic velocities c_1 and c_2 found in [2] in the polar coordinates r and θ of the u_1, u_2 plane

$$\rho c_{1,2}^2 = d + f' \mp (d^2 + 2gd\cos 2\theta + g^2)^{1/2}$$

By virtue of the smallness of g, we obtain for the velocities of quasi-rotational small perturbations c_θ: $\rho c_\theta^2 = f/r + g\cos\theta$, $\rho(c_\theta^-)^2 = f_0/R + g\cos\theta_0$. The angle θ_0 corresponds to the state ahead of the jump. The velocity W of a discontinuity of the same type is found from (3.3)

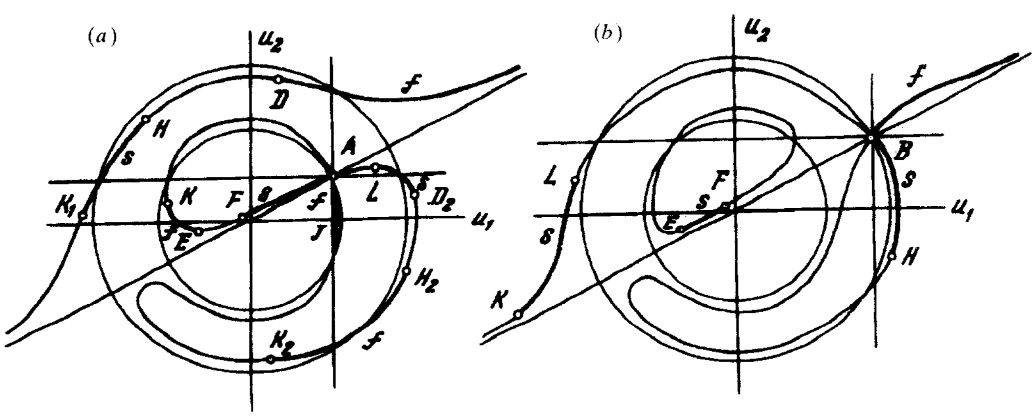

Fig. 4.

$$\rho W^2 = \frac{f}{r} + g\frac{U_1u_2 + U_2u_1 - 2U_1U_2}{U_1u_2 - U_2u_1}$$

or, in polar coordinates

$$\rho W^2 = \frac{f}{r} + g\frac{\sin(\theta+\theta_0) - (R/r)\sin 2\theta_0}{\sin(\theta-\theta_0)} = \frac{f_0}{R} - g\frac{\sin(\theta+\theta_0) - (r/R)\sin 2\theta}{\sin(\theta-\theta_0)}$$

The evolution conditions (2.3) for domains of the shock adiabatic curve close to the circles require that the inequalities

$$\begin{aligned} W^2 - c_\theta^2 &= g\frac{\tfrac{3}{2}\sin(\theta+\theta_0) - \tfrac{1}{2}\sin(3\theta-\theta_0) - (R/r)\sin 2\theta_0}{\sin(\theta-\theta_0)} \leqslant 0 \\ W^2 - (c_\theta^-)^2 &= g\frac{(r/R)\sin 2\theta - \sin(\theta+\theta_0) - \cos 2\theta_0 \sin(\theta-\theta_0)}{\sin(\theta-\theta_0)} \geqslant 0 \end{aligned} \tag{4.2}$$

be satisfied.

It is sufficient to calculate the coefficients of the small g in these formulae at the points of the corresponding circles, that is, to put $r = R$ for quasi-rotational waves ($r = R$ is the circle passing through the initial point wherever it is located) and for the other almost circular branches of the shock adiabatic curve $r = r_B > R$ in Fig. 4(a) and $r = r_A < R$ in Fig. 4(b). We recall here that $c_\theta = c_2$ when $d > 0$ and $c_\theta = c_1$ when $d < 0$ which enables one to distinguish which of the discontinuities are fast and which are slow.

The graphical solution of the equalities (4.2) yields the Jouguet points which act as the boundaries of the evolutionary intervals and are denoted in Fig. 4 by the letters F, K, K_1, K_2, D_1, $D_2(W = c_2^-)$, $L(W = c_1^-)$, E, H, $H_2(W = c_1^+)$, $J(W = c_2^+)$. Investigation of the signs of expressions (4.2) separates out the evolutionary segments in the shock adiabatic curve. They all simultaneously satisfy the requirement that $[S] \geqslant 0$.

In the case of quasi-planar-polarized waves, the analogous sections have been found above in Section 3 for any function $p(u_\alpha)$. Hence, all of the domains of the shock adiabatic curve have been determined where conditions (2.3) and $[S] \geqslant 0$ are simultaneously satisfied. They are picked out by the bold lines in Fig. 4. The fast and slow discontinuities are labelled with the letters f and s, respectively.

We add that the entire investigation can also be used for other functions $f(r)$; for example, with opposite alternation of the convexity and concavity of the graph. In the latter case, if the u_1 and u_2 axes are interchanged, the form of the shock adiabatic curve remains as before as well as the position of the Jouguet points on it. The intervals, where the requirements for evolutionary character and no decrease in the entropy are satisfied, change but are found using the same expressions for c_α, W and $[S]$. The occurrence of additional points of inflection on the graph of f adds new parts of the shock adiabatic curve close to circles of a type such that $r = r_A$ and $r = r_B$.

In particular, the results which have been obtained can be applied to the case when $[U_i]$ are small. They are completely identical to the results obtained previously for low-intensity shock waves [3].

Note that the additivity of the dependence of the elastic potential on the entropy, which has been assumed in (1.3), actually has no effect on the results obtained above referring to rotational waves. The assumption (1.3) means that an entropy change has no effect on the stresses in the medium. If one considers a quasi-rotational wave in which $r \approx R$ then, for sufficiently small g, this assumption can be taken to be valid as a consequence of the smallness of $[s]$. If a jump to a point close to a circle which does not pass through the initial state is considered such as, for example, from point A to a point close to the circle $r = r_B$, then $S = S_B + \Delta S$, where ΔS is small for small g. In the final state, one may then accept equality (1.3), assuming that $F = F(r^2, S)$, $\psi = \rho T_B \Delta S$, $F^- = F(R^2, S_A)$ and analogous equalities for $\partial F/\partial r$.

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